# Optimal Scheduling of Models and Horizons for Model Hierarchy Predictive Control

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***Abstract*— Model predictive control (MPC) is powerful to control systems with non-linear dynamics and constraints, but its computational demands impose limitations on the dynamics model used for planning. Instead of using a single complex model along the MPC horizon, model hierarchy predictive control (MHPC) reduces solve times by planning over a sequence of models of varying complexity within a single horizon. Choosing this model sequence can become intractable when considering all possible combinations of reduced order models and prediction horizons. This paper proposes a framework to systematically optimize a model schedule for MHPC. We leverage trajectory optimization (TO) to approximate the accumulated cost of the robot in closed-loop. We trade off performance and solve times by minimizing the dimensionality of the MHPC problem along the horizon while keeping the approximate closed-loop cost near optimal. The framework is validated in simulation with a planar humanoid robot as a proof of concept. We find that the approximated closed-loop cost matches the simulated one for most of the model schedules, and show that the proposed approach finds optimal model schedules with total horizons that vary between 1.1 and 1.6 walking steps and that transfer directly to simulation.**

## I. INTRODUCTION

### *A. Related Work*

Full-body models can exploit the full dynamic capabilities of the robot [1], but it remains challenging to optimize full-body trajectories in real time for high-dimensional systems like legged robots. Due to the complexity of these systems, roboticists have adopted reduced order models (ROMs) to simplify analysis and control.

For instance, the linear inverted pendulum (LIP) [2] and the spring-loaded inverted pendulum (SLIP) [3] are common ROMs for humanoid locomotion, and have inspired many extensions [4]–[6]. The LIP, the SLIP and the single rigid body (SRB) have been used in MPC to rapidly optimize trajectories, which are typically tracked by a whole-body controller. Despite its utility, this approach can limit performance, as it forces the robot to behave like a ROM that cannot account for whole-body constraints [7].

To find less restrictive ROMs, Chen and Posa [7] proposed a bilevel optimization to synthesize ROMs that minimize the cost incurred by a full-body model. For a chosen ROM parameterization, they find parameters that are least restrictive for a distribution of tasks. The ROM and horizon, however, still need to be chosen by the engineer.

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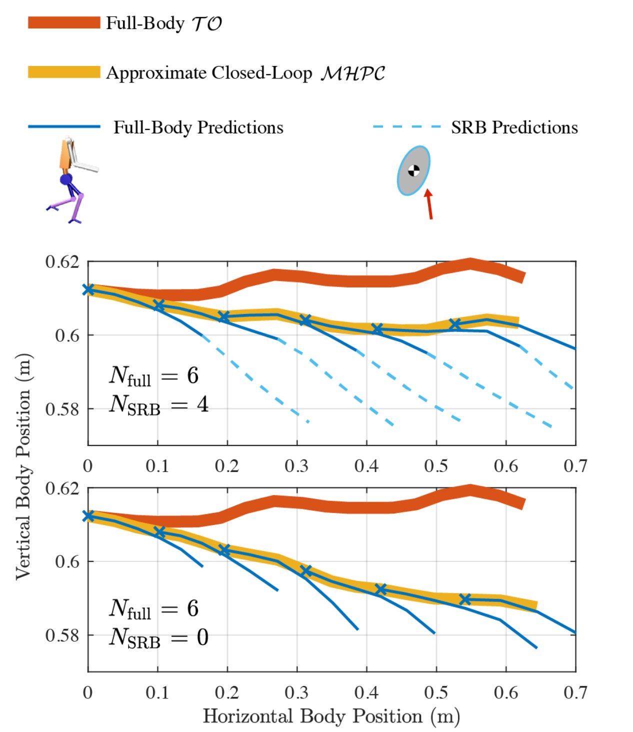


Fig. 1. Body position from full-body ($\mathcal{TO}$) compared to an approximate closed-loop trajectory using ($\mathcal{MHPC}$). The goal of this paper is to optimize the horizons $N_{\text{full}}$ and $N_{\text{SRB}}$ by minimizing the dimensionality of ($\mathcal{MHPC}$) while keeping the closed-loop cost near optimal.

A recent strategy that combines the advantages of ROMs and full-body models is model hierarchy predictive control (MHPC), which consists of planning over a hierarchy of models across the horizon [8]–[10]. Both the performance and solve times of the close-loop controller are affected by the complexity of the models used and their prediction horizon. In previous works, a balance between the number of failures [8], [10] and the average solve time was used to judge the performance of various model schedules. Wang *et al.* [9] used the closed-loop accumulated cost instead of the number of failures. These criteria require running multiple MHPC configurations in simulation to select the one that best trades off solve time and model complexity. This approach, however, becomes intractable when considering the plethora of ROMs available and all possible combinations accross the horizon, each having different equations of motion.

## B. Contribution

The main contribution of this paper is a systematic framework to optimize the model schedule for MHPC. We leverage trajectory optimization to trade off performance and solve time by minimizing the dimensionality of the MHPC controller while keeping the estimated closed-loop cost of the robot near optimal. The approach only requires to derive the equations of motion of the full model, and appropriate constraints to represent various ROMs. We test the approach for a simple walking task, where we solve for the optimal horizons of a full-body model and a SRB model. To validate the approach, we test all possible MHPC configurations in simulation. We show that our method produces a low-resolution approximation of the closed-loop dynamics that enables the use of TO to estimate the closed-loop performance of MHPC. Finally, our TO-based framework solves for model schedules that perform close to a full-model MPC while keeping the solve times low.

# II. BACKGROUND

## A. Trajectory Optimization and Model Predictive Control

Trajectory optimization solves for a trajectory of states and control inputs that minimize the cost along a horizon of $N$ stages while respecting systems dynamics $\boldsymbol{f}(\cdot)$, path constraints $\boldsymbol{g}(\cdot)$ and terminal constraints $\boldsymbol{g}_N(\cdot)$:

$$V^*(\boldsymbol{x}_1) = \min_{\boldsymbol{\mathcal{X}}} \sum_{k=1}^{N-1} c(\boldsymbol{x}_k, \boldsymbol{u}_k) + c_N(\boldsymbol{x}_N) \qquad (\mathcal{TO})$$

subject to

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k) \ \forall \ k \in \{1, ..., N-1\}$$
$$\boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k) \leq \boldsymbol{0} \ \forall \ k \in \{1, ..., N-1\}$$
$$\boldsymbol{g}_N(\boldsymbol{x}_N) \leq \boldsymbol{0},$$

where $V^*(\cdot)$ is the optimal accumulated cost, $\boldsymbol{x}_k$ and $\boldsymbol{u}_k$ are respectively the system state and input at stage $k$, $c(\cdot)$ is the stage cost, $c_N(\cdot)$ is the terminal cost and $\boldsymbol{\mathcal{X}} = \{\{\boldsymbol{x}, \boldsymbol{u}\}_{k=1}^{N-1}, \boldsymbol{x}_N\}$ describes the set of all decision variables: the trajectories of states and inputs.

## B. Model Hierarchy Predictive Control

Instead of planning with a single model, MHPC consists of planning over a hierarchy of models within a single TO, with hybrid dynamics that switch between reduced order models according to a model schedule. In this paper, we define $\boldsymbol{\mathcal{S}} = \{\{\hat{\boldsymbol{f}}, \hat{\boldsymbol{g}}\}_{k=1}^{\hat{N}-1}, \hat{\boldsymbol{g}}_{\hat{N}}\}$ as a model schedule composed of different ROM dynamics and constraints at each stage. MHPC takes the following form:

$$\hat{V}^{\boldsymbol{\mathcal{S}}}(\hat{\boldsymbol{x}}_1) = \min_{\hat{\boldsymbol{\mathcal{X}}}} \sum_{k=1}^{\hat{N}-1} \hat{c}_k(\hat{\boldsymbol{x}}_k, \hat{\boldsymbol{u}}_k) + \hat{c}_{\hat{N}}(\hat{\boldsymbol{x}}_{\hat{N}}) \quad (\mathcal{MHPC})$$

subject to

$$\hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{f}}_k(\hat{\boldsymbol{x}}_k, \hat{\boldsymbol{u}}_k) \ \forall \ k \in \{1, ..., \hat{N}-1\}$$
$$\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{x}}_k, \hat{\boldsymbol{u}}_k) \leq \boldsymbol{0} \ \forall \ k \in \{1, ..., \hat{N}-1\}$$
$$\hat{\boldsymbol{g}}_N(\hat{\boldsymbol{x}}_N) \leq \boldsymbol{0},$$

where the symbol $\hat{(\cdot)}$ denotes variables and functions that relate to ROMs, $\hat{\boldsymbol{\mathcal{X}}} = \{\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{u}}\}_{k=1}^{N-1}, \hat{\boldsymbol{x}}_N\}$ is the set of decision variables. Note the dependency of the dynamics and constraints on stage $k$. The cost accumulated by the robot under the MHPC policy is denoted as $V^{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}_1)$, where the absence of $\hat{(\cdot)}$, emphasizes that the cost is incurred by the actual robot when closing the loop.

# III. PROBLEM STATEMENT

The goal of this paper is to propose a systematic approach to find a model schedule $\boldsymbol{\mathcal{S}}$ that minimizes solve time while bounding the closed-loop performance. As a proxy for solve time, we minimize the dimensionality in ($\mathcal{MHPC}$). Thus, we consider the model schedule $\boldsymbol{\mathcal{S}}$ to be optimal when

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^* = \arg\min_{\boldsymbol{\mathcal{S}}} |\hat{\boldsymbol{\mathcal{X}}}| \qquad (\mathcal{MS})$$

subject to

$$\frac{V^{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}_1) - V^*(\boldsymbol{x}_1)}{V^*(\boldsymbol{x}_1)} \leq \epsilon,$$

where the cost penalizes the number of decision variables (i.e. the dimensionality of ($\mathcal{MHPC}$)). The term in the inequality represents the relative cost error (RCE), which must be within some chosen bound $\epsilon$.

# IV. IMPLEMENTATION

The problem ($\mathcal{MS}$) is a bilevel optimization, with ($\mathcal{MHPC}$) in its inner loop. Solving it requires computing:

1) the optimal cost $V^*(\boldsymbol{x}_1)$ from ($\mathcal{TO}$) for the full robot
2) the solution to ($\mathcal{MHPC}$) and induced policy for multiple $\boldsymbol{\mathcal{S}}$
3) the closed-loop cost $V^{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}_1)$ for multiple $\boldsymbol{\mathcal{S}}$.

In this work, we use trajectory optimization to approximate these three terms.

## A. Full-Body Trajectory Optimization

To estimate $V^*(\boldsymbol{x}_1)$, we formulate TO with a direct transcription method with full-body dynamics.

## B. Parameterized MHPC formulation

The dynamics and state/input constraints in the MHPC need to be changed depending on the model used. To achieve this, the model schedule $\boldsymbol{\mathcal{S}}$ is defined as the set of binary parameters

$$\boldsymbol{\mathcal{S}} = \Big\{ s_{m,k} \in \{0, 1\} | \sum_{m \in \mathcal{M}} s_{m,k} = 1 \ \forall \ k \Big\}, \qquad (1)$$

where the constraint in (1) ensures that only one model is active at each stage $k$. Each element $s_{m,k}$ is used to activate the dynamics and constraints of a ROM $m \in \mathcal{M}$ at stage $k$, where $\mathcal{M}$ is a finite set of candidate models.

Instead of deriving the equations of motion for each possible reduced-order model, the full-body dynamics are used along the entire horizon and are constrained to enforce the behavior of various reduced order models at each stage. In practice, the big-M formulation is used to activate and deactivate the constraints.

### C. Closed-loop Cost Approximation

To approximate the closed-loop cost $V^{\boldsymbol{\mathcal{S}}}$ in ($\mathcal{MHPC}$), the full-body model is used at the first and second timesteps. This restriction allows to directly treat the first timestep of ($\mathcal{MHPC}$) as a simulation step. As a result, the solution to ($\mathcal{MHPC}$) approximates the closed-loop dynamics of the robot under the MHPC policy using TO:

$$\{\boldsymbol{u}_k, \boldsymbol{x}_{k+1}\} \subset \arg\min_{\hat{\boldsymbol{\mathcal{X}}}} \hat{V}^{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}_k). \tag{2}$$

With (2), a full-body trajectory over a horizon of $N$ timesteps is generated (as in Fig. 1), for which the accumulated cost $V^{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}_1)$ is evaluated. Hence, the inequality in ($\mathcal{MHPC}$) can be evaluated for every $\boldsymbol{\mathcal{S}}$ and we can find $\boldsymbol{\mathcal{S}}^*$ that minimizes the dimensionality.

## V. RESULTS

### A. Experimental Setup

We implement our approach for a fixed walking gait. The order of each model is fixed (full-body model first, followed by the SRB and the void models), and we solve for the horizons of the full and SRB models $N_{\text{full}} \in \{2, ..., N\}$ and $N_{\text{SRB}} \in \{0, ..., N-2\}$. We refer to a given model schedule by the tuple $(N_{\text{full}}, N_{\text{SRB}})$.

We implement our proposed approach in MATLAB with a 13-DoF planar biped. All optimizations are formulated with CasADi and solved with the non-linear interior-point solver in Knitro 13.1. To validate our approach, MHPC is simulated at 100 Hz for all model schedules using ode45.

### B. Approximate Closed-Loop Trajectories

Repeatedly solving (2) generates trajectories that approximate the closed-loop behavior of ($\mathcal{MHPC}$). This allows us to directly leverage TO to evaluate the closed-loop accumulated cost for a given model schedule. Fig. 1 compares body trajectories from ($\mathcal{MHPC}$) with the approximate closed-loop for two model schedules (6,4) and (6,0). With (6,0), the approximate closed-loop trajectory is farther away from the full-body optimized trajectory. For (6,4), the additional SRB horizon reduces the gap between the full-body TO and the approximate closed-loop.

### C. Relative Cost Error

Fig. 2a illustrates the relative cost error (RCE) evaluated from the approximated closed-loop trajectory for each model schedule. There is a sharp change in RCE, at $N_{\text{full}} = 5$, above which many model schedules achieve low RCE and successful walking in simulation. The diagonal lines indicate isolines for the total horizon $N_{\text{full}} + N_{\text{SRB}}$. Note that the RCE is high below 1 walking step and rapidly drops when planning above 1 walking step.

### D. Optimal Model Schedules

Fig. 2b shows the dimensionality of each model schedule, which we wish to minimize. The boundaries delimit the feasible sets of model schedules for $\epsilon = 0.1$ and $\epsilon = 0.001$, as determined by the inequality constraint in ($\mathcal{MS}$). Within these sets, the corresponding optimal schedules (4,7) and (8,3) are marked by "+" signs.

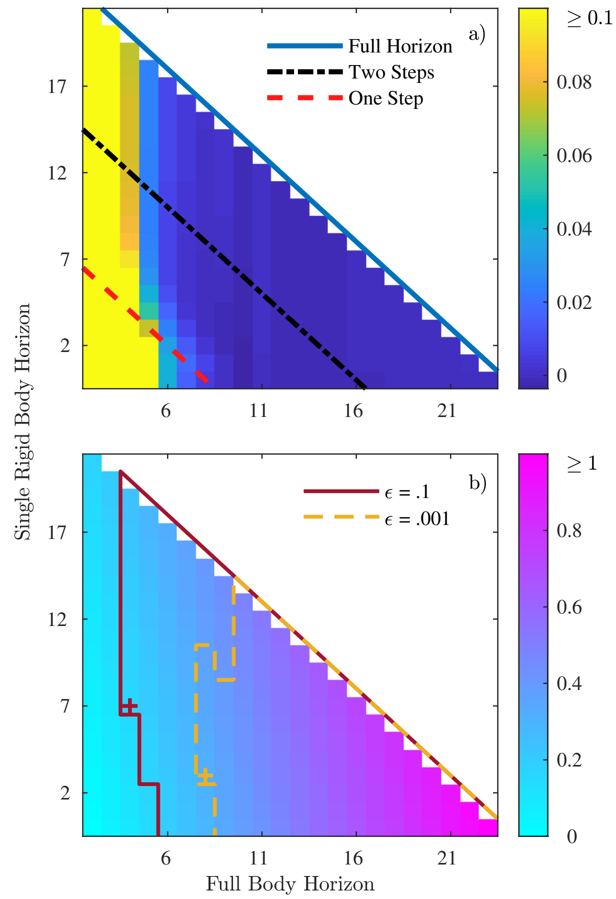


Fig. 2. a) Relative cost error and b) normalized dimensionality of ($\mathcal{MHPC}$) for all possible model schedules. The three diagonal lines in a) are isolines for the total horizon $N_{\text{full}} + N_{\text{SRB}}$ that highlight horizons of one walking step, two walking steps or the full horizon ($\hat{N} = 23$). In b), for each choice of $\epsilon$, the lines delimit the feasible sets of model schedules from ($\mathcal{MS}$) and the "+" signs mark the optimal solution for each $\epsilon$.

While we formulate ($\mathcal{MHPC}$) with a total of 23 stages, the proposed method finds optimal model schedules that substantially truncate the total horizon by making use of the void model. We solved ($\mathcal{MS}$) for 100 values of $\epsilon$ ranging from 0.01% to 3%, and the optimal model schedules always give total horizons $N_{\text{full}} + N_{\text{SRB}}$ between 10 and 14 stages, corresponding to between 1.1 and 1.6 walking steps. This result is in line with previous work where it was shown that planning beyond two walking steps does not provide a significant advantage [11].

### E. Relative Cost Errors from TO and Simulation

The RCE obtained with the TO-based approximations vary similarly to the RCE from the simulations. To illustrate this, we normalize both RCEs from TO and simulation by their respective maxima, and compute their difference. As shown in Fig. 3, the difference is close to 0 for most model schedules, indicating that the shapes of each RCE closely match. Therefore, the closed-loop performance of ($\mathcal{MHPC}$) is well approximated by the TO-based approach.

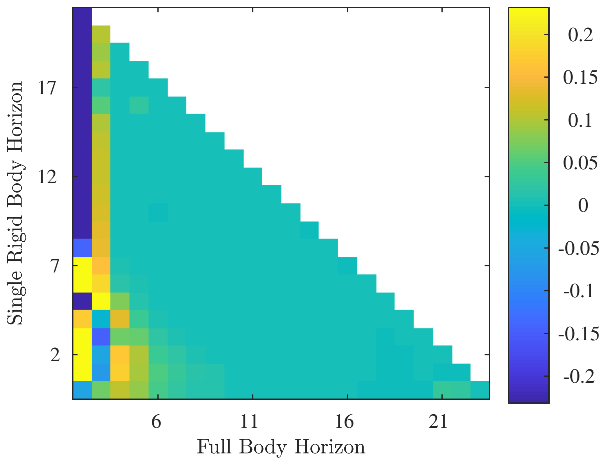


Fig. 3. Normalized error of accumulated cost between the approximate closed-loop ($\mathcal{MHPC}$) and simulated ($\mathcal{MHPC}$) trajectories for all model schedule possibilities. The absence of gradient for the error in most of the schedules indicate that the approximation is a good proxy for running ($\mathcal{MHPC}$) in a more accurate simulation.

## VI. CONCLUSION AND OUTLOOK

We proposed a framework to systematically optimize the model schedule to use along the planning horizon of MHPC. We validated the approach with a proof of concept, where we solve for optimal horizons of full-body and SRB models. The framework leverages trajectory optimization to estimate the closed-loop cost incurred by the robot under a MHPC controller. The method finds model schedules that minimizes the dimensionality of the MHPC controller while keeping the accumulated cost incurred by the robot close to optimal.

In this work, the number of possible model schedules was sufficiently small to solve ($\mathcal{MHPC}$) using enumeration. Future work should make use of more efficient algorithms to scale the approach for a wider variety of reduced order models. One possible approach is to solve ($\mathcal{MS}$) using gradient-free evolutionary algorithms or with a bilevel mixed-integer non-linear solver with ($\mathcal{MHPC}$) in the inner loop. For the latter, the proposed MHPC formulation with integer variables could be used directly, and recent work on optimization based-dynamics could be leveraged [12].

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